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THE TEACHING OF BEGINNING GEOMETRY

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It seems necessary before presenting the main points of this paper to discuss, briefly, some of the underlying principles on which they are based. I ask your indulgence, therefore, if I appear to go somewhat far afield in my search for an adequate setting.

Geometry is essentially the study of spatial relations. "Of all forms of relational consciousness its concepts are the most directly perceived and familiar facts in human experience," says the philosopher, Ernest Mach. "The only other relational consciousness which is anything like as familiar is time, and these are the two universal forms in which all sensory facts appear in experience. . . . The first geometrical knowledge was acquired accidentally and without design by way of practical experience, and in connection with the most varied employment." Hence, in its fundamental aspects, the science of geometry differs from the science of physics only in the respect that mental experiment can be performed therein with far simpler experiences and with such as have been more easily and almost unconsciously acquired. In the illuminating language of Poincaré, "Geometry is not true, it is advantageous. . . . By natural selection our mind has adapted itself to the conditions of the external world." Our notions of space are, therefore, "rooted in our physiological organism"; so that, "without the least artificial or scientific assistance we acquire abundant experience of space. We can judge approximately whether rigid bodies which we perceive along side one another in different positions at diffrent distances will, when brought successively into the same positions, produce approximately the same or dissimilar space sensations. We know fairly well whether one body will coincide with another, whether a pole lying on the ground will reach to a certain height, etc." "It is extremely probable," says Mach, "that the experiences of the visual sense were the cause of the rapidity with which geometry developed"; for, "visual space forms the clearest, precisest and broadest system of space sensations."

"From the historical point of view," says Judd, in his *Psychology of High School Subjects,* "one would expect geometry to come early in the course, in view of the fact that it matured earlier than algebra. The fact is," continuing the quotation, "its very perfection served to take geometry into the highest schools. In the University of Alexandria the results of Greek studies of space were put into logical form by Euclid. This logical form was also borrowed from Greece, where Aristotle had evolved that perfect system of syllogistic logic which dominated the whole period of mediæval thought. A complete geometry expressed in perfectly logical form became one of the highest courses of study. Studies of space were no longer of the primitive type that had grown up among the early Greeks." So well did the Greeks, in the critical period of Greek mathematics, attain their ideal—"to systematize the knowledge of their province for professional and scholarly contemplation and for purposes of individual survey, to separate what was directly cognizable from what was deducible, and to throw into distinct relief the thread of deduction," that today geometry is no longer recognized as a separate science, distinct from mathematics, as is physics or chemistry.

It appears manifest, therefore, that the traditional difficulties of beginning geometry are not traceable to the spatial conceptional material, but must be sought elsewhere. Poincaire, in his essay on "The Philosophy of Science," offers the following corrective to the difficulties: "Zoologists maintain that the embrionic development of an animal recapitulates in brief the the whole history of its ancestors throughout geologic time. The teacher should make the child go over the path his father trod; more rapidly but without skipping stations. For this reason," he adds, "the history of science should be our first guide." The same idea is tersely stated by Mach as follows: "An idea is best made the possession of the learner by the method by which it has been found." From Felix Klein we have the following advice: "In teaching it is not only admissable, but absolutely necessary to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them. . . . If, too, much emphasis is placed on rigor at the beginning a large part of the subject will remain unintelligible."

In the light of this wholesome advice, and in order to give our setting an historical background, let us trace, briefly, the development of geometry from antiquity. The Greeks inherited the nucleus of classified geometric knowledge from the Egyptians. The total amount of this information consisted of a few practical rules, some of which were little better than mere guesses, for example, their rule for finding the area of a quadrilateral, namely, the product of the arthmetical means of the opposite sides. Perhaps the most celebrated rule which the Egyptians handed down to the Greeks was the one for measuring off a right angle by means of constructing a right triangle whose sides are in the ratio 3:4:5. Regarding geometry as a science, the Egyptians had not even made a beginning, and, says the historian, "there are reasons for thinking that Egyptian knowledge remained virtually stationary for at least two thousand years prior to their contact with the Greeks."

"The founder of the earliest Greek school of mathematics and philosophy was Thales, one of the seven sages of Greece, who was born about 640 B. C. at Miletus, and died in the same town about 550 B. C. We cannot form an exact idea as to how Thales presented his geometrical teaching," says the historian. "We infer, however, from Proclus that it consisted of a number of isolated propositions which were not arranged in a logical sequence, but that the proofs were deductive, so that the theorems were not a mere statement of an induction from a large number of special instances, as probably was the case with the Egyptian geometricians. The deductive character which he thus gave to the science is his chief claim to distinction. The following comprise the chief propositions that can now with reasonable probability be attributed to him; they are concerned with the geometry of angles and straight lines.

"(1) The angles at the base of an isosceles triangle are equal. . . . Proclus seems to imply that this was proved by taking another exactly equal isosceles triangle, turning it over and then superposing it on the first—a sort of experimental demonstration.

"(2) If two lines cut one another, the vertically opposite angles are equal. Thales may have regarded this as obvious, for Proclus adds that Euclid was the first to give a strict proof of it.

"(3) A triangle is determined if its base and base angles are given.

"(4) The sides of an equiangular triangle are proportional.

"(5) A circle is bisected by any diameter. This may have been enunciated by Thales, but it must have been recognized as an obvious fact from the earliest times.

"(6) The angle subtended by a diameter of a circle at any point in the circumference is a right angle. . . . This appears to have been regarded as the most remarkable of the geometrical achievements of Thales, and . . . it has been conjectured that he may have come to this conclusion by noting that the diagonals of a rectangle bisect each other, and that therefore a rectangle can be inscribed in a circle. If so," continues the historian, "and if he went on to apply proposition (1), he would have discovered that the sum of the angles of a right-angled triangle is equal to two right angles, a fact with which it is believed he was acquainted. It has been remarked that the shape of the tiles used in paving floors may have suggested these results. . . ." Commenting further, the historian suggests "that it is possible he was also aware, as suggested by some modern commentators, that the sum of the angles of any triangle is equal to two right angles. As far as equilateral and right-angled triangles are concerned, we know from Eudemus that the first geometers proved the general property separately for three species of triangles, and it is not unlikely that they proved it thus. The area about a point can be filled by the angles of six equilateral triangles or tiles, hence the proposition is true for an equilateral triangle. Again, any two equal right-angled triangles can be placed in juxtaposition so as to form a rectangle, the sum of whose angles is four right angles; hence, the proposition is true for a right-angled triangle. Lastly, any triangle can be split up into the sum of two right-angled triangles by drawing a perpendicular from the biggest angle on the opposite side, and therefore again the proposition is true. The first of these proofs is included in the last, but there is nothing improbable in the suggestion that the early Greeks continued to teach the first proposition in the form above given."

I have quoted the historical comments here at some length because of their relevancy to primitive deductive methods. It is evident that the results were arrived at by a kind of "sensuous mental experimentation with figures.'' Hence, if the history of geometry is to be a trustworthy guide, we should be led to indorse the ideas of Henrici, that in the beginning the teacher should put the subject in such a manner that (the student) may realize the geometrical contents of the.propositions as properties of space through actually seeing their truth by the mental or physical inspection of figures, instead of being convinced of their truth by a long process of logical reasoning.

Following Thales, the next great Greek geometer to advance the science was Pythagoras who flourished from about 569 B. C. to about 500 B. C., and " 'who,' says Proclus, quoting from Eudemus, 'changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated the theorems in an . . . intellectual manner.' '' Says Ball, ''Pythagoras probably knew and taught the substance of what is contained in the first two books of Euclid about parallels, triangles, and parallelograms, and was acquainted with a few other isolated theorems including some elementary propositions on irrational magnitudes; but it is suspected that many of his proofs were not rigorous, and in particular that the converse of a theorem was sometimes assumed without proof.''

A point worth noting in this connection is the fact that Pythagoras and his contemporaries, mentally keen though they were, were guilty of ignoring many logical refinements which we are sometimes reluctant to permit even during the first few weeks on the part of rather young pupils.

Following Pythagoras was a period of about 200 years when Greek philosophy was rife with sophists and paradox jugglers. These latter, says Ball, ''made the Greeks look with suspicion on the use of infinitesimals,'' and may we not also add, probably influenced Euclid in his evasive use of motion, when a free use of the latter would have added much to the clearness of his exposition.

Undoubtedly the two most influential factors in bringing about the extreme formalism of Greek geometry were the two great philosophers, Plato and Aristotle, who lived in the century preceding Euclid. Primarily not mathematicians in the sense of being original investigators, ''the latter,'' says Ball, ''was chiefly concerned with mathematics and mathematical physics as supplying illustrations of correct reasoning; and to the latter is due that subsequent geometricians began the subject with a carefully compiled series of definitions, postulates, and axioms.''

Commenting on these historical notes, we see that unconscious intuition played an extremely important part in early Greek geometry. ''Indeed,'' says Klein, ''history reveals in every case that unconscious intuition has been especially active during the genesis of every important advance in the development of mathematics.'' We see that in its early stages, the process of deduction was little more than a sort of visual inferential process based on placing figures in certain favorable positions, either by accident or design. Pertinent to this point is the following excerpt from his essay on ''The Philosophy of Science,'' by Poincaré:

''The principal aim of mathematical teaching is to develop certain faculties of the mind, and among them intuition is not the least precious. It is through it that the mathematical world remains in contact with the real world, and if pure mathematics could do without it, it would always be necessary to have recourse to it to fill up the chasm which separates the symbol from reality. For the pure geometer himself, this faculty is necessary; it is by logic one demonstrates, by intuition one invents. To know how to criticize is good, to know how to create is better. . . . Logic tells us that on such and such a way we are sure not to meet an obstacle; it does not say which way leads to the end. For that it is necessary to see the end from afar, and the faculty which teaches us to see is intuition. Without it a geometer would be like a writer who should be versed in grammar but had no ideas. Now how could this faculty develop, if as soon as it showed itself we chase it away before knowing the good of it? . . . In the exposition of first principles . . . it is necessary to avoid too much subtility; there it would be most discouraging and moreover useless. We cannot prove everything and we cannot define everything; and it will always be necessary to borrow from intuition; and what does it matter whether it be done a little sooner or a little later, provided that in using correctly premises it has furnished us we learn to reason soundly. . . . The essential thing is to reason soundly on the assumptions admitted.''

But intuition is not exact, and this is the point where the hypercritical logician inserts his objections. He abhors such statements as ''geometry is concrete,'' unmindful of the fact that we cannot think except in terms of imagery which has a degree of objectivity, for ''only the actually visualized or imaged figures can tell us what particular concepts are employed in a given case.'' Klein illustrates this graphically as folows: ''Newton,'' he says, ''assumes without hesitation the existence in every case of a velocity of a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivatives. Thus,'' he continues, ''when thinking of a point, we do not think of an abstract point, but for something concrete for it. . . . In imaging a line, we do not picture to ourselves 'length without breadth,' but a strip of certain width. Now such a strip has, of course, always a tangent; that is, we can always imagine a strip having a small portion (element) in common with the curved strip. . . . The definition in this case is regarded as holding only approximately, or as far as may be necessary.''

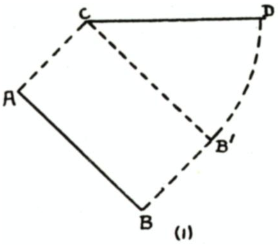
Reverting again to the historical references in order to complete the review, we note that the extreme formalism of Greek geometry was a comparatively late development, reaching its present state of perfection fully 300 years after the first school of Greek mathematics; and largely induced by the influence of professional logicians, not mathematicians, whose principal aim was to produce philosophers, and not geometers.

Therefore, to be in harmony with the lessons of history, let us begin geometry in a simple, natural manner, giving free play to the intuition. Let us postpone formal methods until the mind of the pupil is familiarized and matured by a sufficient frequentation with informal geometric reasoning. Let us not begin our subject by extracting all objectivity from the notion of planes, lines and points, but let us tacitly regard the planes as ''corporeal sheets,'' the line as ''a thread vanishingly small in thickness,'' and a point as a small ''corporeal space.'' ''Nothing prevents our idealizing in the usual manner these images at the proper time by simply leaving out of account the thickness of the sheets and threads.'' For our introduction let us present a schematized progressive development which shall lead the pupil gradually, by easy stages, to an appreciation of formal deductive methods. In this preparatory course, let us avoid the technical language of formal logic, such as hypothesis and conclusion; and above all, let us not insult the intelligence of our pupils, or mystify them "by telling them that they do not understand what they think they understand, and that they must prove what seems to them evident." In this progressive development, let us develop the properties of the straight line and the circle together, because each throws light on the other, and because these properties can very frequently be inferred immediately by a mental inspection of the figure; and, thirdly, because many of the propositions can be immediately applied in geometric drawing where they will seem useful to the student.

Let us, therefore, begin geometry with ruler and compasses, forth with. From the tests of the straight edge will arise the fundamental properties of the straight line; the elementary properties of the circle will follow quite as readily. These being obvious intuitions, may be passed over with simple acquiescence on the part of the pupil. Simple exercises in drawing may then be given, always having in mind the encouraging of thought processes, chiefly intuitional at this stage, and not simply mechanical manipulation. I prefer to stress and encourage independent thinking from the first day, for "the initial method will largely determine the subsequent mental attitude of the pupil toward a subject, no matter what may afterward be done."

A few simple examples with concrete settings, and calling for the exercise of clear visualization will be appropriate. As an example, I mention the following which interested me greatly when I was a boy, and which was contained in an old arithmetic, I think it was Ray's: "A spider in an upper corner of a square room of given dimentions observes a fly in the lower diagonally opposite corner. If the spider proceeds to attack his prey by the most direct route, traveling on the walls and ceiling, or floor, what will be his path?" With some discussion followed by a few suggestions, including the hint that the room may be represented by a cardboard box, a solution will be forth coming. The solution which consists in flattening out the walls of the room into the plane of the floor, and joining the original position of the spider with that of the fly will not only be convincing, but it will provide the opportunity to drive home the point that a very obvious property of the straight line is the key to the solution which is not at all obvious. This will promote respect for summaries of obvious properties of figures.

Our next important consideration is that of motion. Without doubt motion should play a larger part, and should receive much more emphasis in elementary geometry than it has in the past. In his essay on "The Philosophy of Science," Poincaré points out that "geometry (viewed from the standpoint of higher analysis) is the study of a group; that of the motion of solid bodies"—as a side light bearing on another point, he says, "How define this group then without moving some solids." In another vein, Mach remarks, "The study of the movements of rigid bodies, which Euclid studiously avoids and only covertly introduces in his principle of congruence, is to this day the device best adapted to instruction in elementary geometry." The notion of congruence is of course inseparable from the idea of motion. Instead of slurring over the motions involved in superposition, they should be stressed. All the more so because intuition plays a big part here, and, if we do not chase it away, referring to the admonition of Poincaré, it will help us over most of the difficulties. The real difficulties underlying these proofs as usually given, are due to the logical technique employed, and which ought to be postponed, and to certain subtilities implied in the obvious properties of the straight line. If, however, the motions are stressed, the logical technique is postponed, and subtle refinements are kept in the back ground, a clear comprehension of the basic principle of congruence will result. If the teacher's preliminary questions are skilfully formed the logical difficulties may be avoided. Thus, after having discussed the three kinds of motion; sliding motion or translation, rotation about a point, and revolving about a straight line as an axis, the teacher may safely ask the following questions: How may the line segment *AB* be brought into coincidence with the equal segment *CD*, employing several examples in which *AB* and *CD* have various relative positions? How may a circle be brought into coincidence with a congruent circle? How may an arc of a circle be brought into coincidence with an equal arc? How may the letter *L* be brought into coincidence with the inverted *L*? And so on. All of these questions will be answered intuitively with little hesitation. If, however, one asks why the segment *AB* coincides with the segment *CD* after *A* falls on *C* and *B* falls on *D*, he will get no answer, or if he does get an answer, it will probably be because the segments are equal. This being an intuitive fact, the student can at this time have no reason to doubt it; to doubt intelligently, "one must know why one doubts." In other words, the fundamental property of the straight line on which the answer depends is a useless refinement at this stage of progress.



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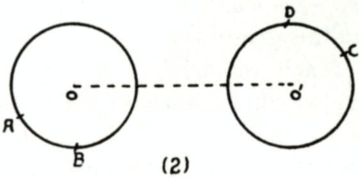

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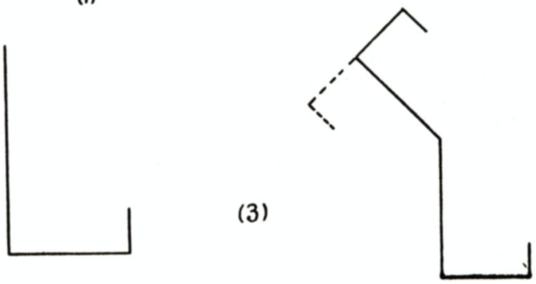

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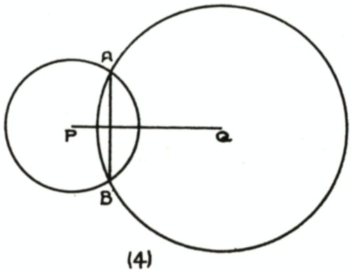
Thus, referring to the illustrations (see Fig. 1) *AB* is brought into coincidence with *CA*, by first sliding *AB* along the line *AC* until *A* coincides with *C*; then by rotating *AB* about *C* as a center. That is, a line segment can always be brought into coincidence with an equal segment, in the same plane, by a slide followed by a rotation. Similarly, the circle *O* may be brought into coincidence with the circle *O′* by sliding the former along the line *OO′* until the centers coincide; also the arc *AB* may be brought into coincidence with the equal arc *CD* by a slide (that is, by bringing the circles into coincidence) followed by a rotation.

Likewise, it is easily visualized that the upright *L* may be brought into coincidence with the inverted *L* by applying consecutively the three types of motion—sliding, rotating and turning (or revolving) motion.

We are now in a position to introduce a principle which will further our schematization and at the same time parallel, in a general way, the methods employed by the contemporaries of Thales—the principle of symmetry. "The circle being the embodiment of symmetry," adopting the phraseology of Henrici, by studying its relations to the straight line in certain favorable positions, we shall be enabled to derive a variety of useful properties by a simple mental inspection of figures.

An ink blot thrown on a scrap of paper which latter is folded through the blot while the ink is still wet will instantly command the attention of the class. There results a grotesque figure to be sure, the form of a beetle's shadow; nevertheless, it exhibits the desired properties, and it is a form which the pupil recognizes as widely distributed in nature. A little discussion will bring out the obvious properties of a symmetric figure.



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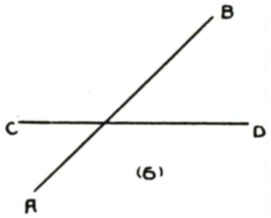

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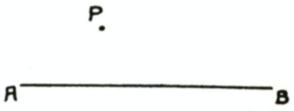

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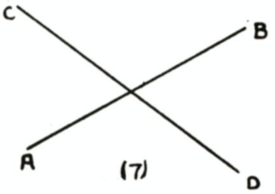

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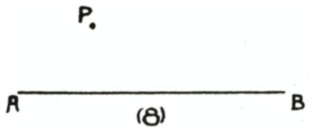
Having defined symmetry with respect to an axis, the fact that a circle is symmetric with respect to any diameter follows immediately as an intuitive fact. It follows not less clearly that two intersecting circles are symmetric with respect to their line of centers. Whence (see Fig. 4), by a mental inspection of the figure we infer immediately that the common chord is bisected by the line of centers. A judicious line of questions will now readily lead to the discovery of the usual method of bisecting a line segment with compasses and ruler. Following the above, a variety of simple exercises may be given, such as: (Fig 5) Draw a straight line AB and an external point P; find a point Q which corresponds to P in the symmetric figure of which AB is an axis. (Fig. 6) Draw two straight lines AB and CD; find a straight line which corresponds to AB in the symmetric figure of which CD is the axis. (Fig. 7) Draw two intersecting straight lines AB and CD; find the axis of symmetry of the figure in which AB and CD are corresponding lines. This, of course, is the familiar problem, ''To bisect an angle,'' expressed in different language; when it appears later in the usual form its mastery by the pupils will require little effort. Simple problems based on architectural drawing may also be assigned to provide concrete applications.

The solution of the first problem, referring to Fig. 5, results from drawing two circles passing through P and having their centers on AB.

In the second problem, Fig. 6, it is obvious that the line corresponding to AB must pass through O, the point where AB cuts CD. The solution, therefore, depends on obtaining a second point lying on the required line. Hence, taking any point on AB, and repeating the construction of problem 1, the solution is obtained.

If now we define perpendicular to a line as a line symmetric to the former as an axis, we may easily solve the usual problems for constructing perpendiculars from a given point to a given straight line. After a few well chosen questions referring to the symmetry of two intersecting circles, the pupils will readily discover or invent the constructions (see Figs. 6 and 9). Moreover, by a simple inspection of the figure resulting from the symmetry of two intersecting circles, we easily infer the following important proposition: ''Every point on the perpendicular bisector of a line segment is equally distant from the end points of the segment'' (Fig. 10). This proposition enables us to find the center of a circle when three points on the circle are given, and other related problems. The student who takes part in solving problems of this kind begins to have awaken within him an appreciation of, and for, geometric reasoning. To foster this growing spirit, I habitually propose a problem in this connection (there are doubtless better ones) whose solution is not at all obvious, but the proof of which will be welcome, and which will carry real conviction: In Fig. 11, *A* represents the position of a house, *B* that of a barn, and *MN* the edge of a river. Required the position of a pumping station, located on the river bank, so that the water can be piped to both house and barn with the least length of pipe. The essentials of the problem will be recognized as being the same as those of the familiar



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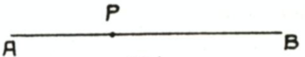

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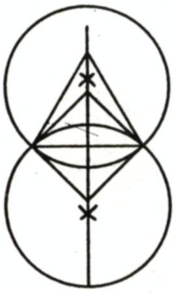

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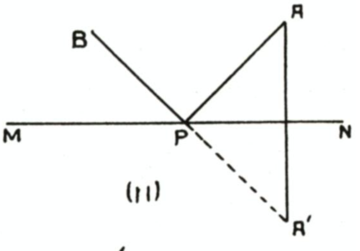

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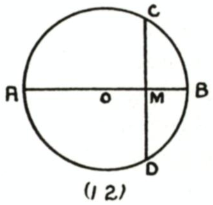

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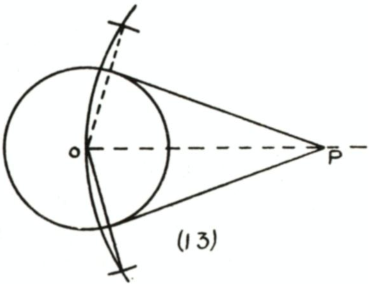

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problem concerning the reflection of light—a more objective setting being substituted. The solution of the problem involves nothing more than the proposition concerning the perpendicular bisector of a line segment, just derived, and the proposition that the shortest path between two points is the line segments joining them. The problem is, of course, too difficult to assign for independent work; however, it yields readily to solution under the eyes of the teacher. The only object in solving it with the pupils is to cultivate an appreciation for a geometric proof differing from a direct inference. This I think it does.

The limits of this paper forbid my developing in too great detail a complete course leading up to formal methods. Suffice it to say, that other constructions may be developed in an analogous manner. Thus, by carrying out in detail the analysis of the motions by which an arc is brought into coincidence with an equal arc, the discussion invoked leads naturally and readily to the relations implicating the corresponding chords and central angles. It is then but a step, and intuition will lead the way to the solution of the problem, to construct an angle equal to a given angle.

The developments leading up to the construction of tangents to a circle will serve to give variety to the illustrations, and may perhaps merit a more detailed description. Consider any circle with center *O* (Fig 12). Draw any diameter *AB*, and a chord *CD* perpendicular to *AB*. From the symmetry of the figure, remembering that a perpendicular to a straight line is symmetric with respect to the latter as an axis, we infer immediately by an inspection of the figure that the chord is bisected by the diameter. The obvious inference that a diameter which bisects a chord is perpendicular to it, and that the perpendicular bisector of a chord passes through the center of the circle will also be noted and summarized. We are now prepared to consider the notion of a tangent. Observing that a straight line whose distance from the center is greater than a radius has no point in common with the circle, and that a straight line whose distance from the center is less than a radius cuts the circle in two points, we are led to inquire as to what must be the relative position of a straight line with respect to the circle, when its distance from the center is exactly equal to the radius. The reasoning may then be given that, theoretically, the line can have but one point in common with the circle. The reasoning will be convincing, and it will be welcome, because it proves something that baffles the visual sense. The method for constructing a tangent to a circle at a given point on the circle will be obvious.

The method for constructing a tangent to a circle from a point outside the circle can then be readily developed. Analyzing the problem, we note that the required line must be perpendicular to the radius drawn to the point of contact. Leading questions referring to the obvious properties of the figure studied at the beginning of the topic bring out the details of the construction (Fig. 13), which consists in describing a circle passing through the center of the given circle and having its center at the given external point *P*. The point of contact will then be where the chord from *O*, and equal to the diameter of the given circle, intersects the latter. It is scarcely necessary to remark that the usual properties implicated in the figure resulting from the construction of the tangents to a circle from an external point may be inferred immediately from symmetry by a simple inspection of the figure.

The topic on tangents provides a variety of exercises, both concrete and abstract, for independent construction work by the pupils. Exercises are also available for making simple inferences, other than from visual inspection of figures. As examples of the latter may be noted the following: "If two circles are tangent to each other, the tangents drawn to them from any point of their common tangent are equal." "If two circles are tangent to each other, externally, their common interior tangent bisects their common exterior tangent."

Informal work of this kind could be prolonged indefinitely. It is a matter of experience and judgment on the part of the teacher as to how much time may be profitably spent in this manner. But let there be no mistake about it, time is not wasted by alloting a reasonable period for this sort of a preparation, and there should be no more failures than in the average of other academic subjects.

Let us suppose that three weeks have now elapsed, and that we are prepared to adopt formal methods of procedure. Let us not, however, fall into the error of thinking that we can begin formal geometry in the technical sense of the term; that is not possible in the high school, least of all at the end of only three weeks of an informal introduction. We can, of course, adopt formal language, but we can only approximate formal methods of deduction. Some discussion of logical terms and the nature of formal statement of proof will be necessary. Having covered this ground, amplifying the explanations with illustrations chosen from the material developed informally, we may proceed with the topic of congruent triangles. I prefer to approach the topic thus:

Given certain dimensions, construct a triangle.

(1) When two sides and the included angle are given.
(2) When two angles and the included side are given.
(3) When the three sides are given.

(The pupils are, of course, familiar with the methods for constructing equal angles, and with the use of the protractor, for that matter.)

Then, having constructed two triangles with two angles and the included side of the one equal, respectively, to two angles and the included side of the other, I provoke a discussion as to how either triangle may be brought into coincidence with the other, arranged in various relative positions. The motions are analyzed thus: Slide triangle ABC along the line AA' until A coincides with A'; then rotate triangle ABC about A' as a center until AB coincides with its equal $A'B'$; then (assuming that we are dealing with the most general relative positions) turn triangle ABC over on $A'B'$ as an axis. When AC will fall on $A'C'$, and BC will fall on $B'C'$. The reason for the latter being correctly given, the conclusion that C falls on C' is reached and justified, visually by the pupils, and logically by the teacher. Theorem (1) having been similarly treated, the formal statements of these proofs as presented in the text are discussed. The pupils are made to see that, save for the language, the proofs are essentially the same.

Henceforth, for some little time, facility in formal statement of proof is the immediate aim. The topic of congruent triangles, as every experienced teacher knows, furnishes the very best drill for this purpose; not by reading or memorizing the standard propositions exhibited in the text, but by proving simple exercises depending on the basal theorems proved above. For, as has been well said by Schulze, "the solution of a large number of simple exercises will enable the student to understand clearly and to appreciate model demonstrations."

Time forbids further detailed discussion of methods of presentation. In passing, however, I may remark, that little text book advance should be made until the pupils are fairly efficient in proving simple exercises of this kind. In this connection, I wish to remark in a general way, that much harm may be done in the beginning by over-emphasizing the recitation method of teaching, especially if the teacher exacts fluent and complete demonstrations. Thus to expect a pupil early in the course to study an average text book demonstration out of class; then on the morrow put the figure on the board, complete in every detail, and, with pointer in hand, rise and give the entire proof, is to approach geometry by the wrong method. The results cannot but be fatal to a large percentage of the class. If one assigns a proposition for home study, early in the course, a much better practice in recitation is to put the figure on the board, without auxiliary construction lines; then, writing the essential steps on the board systematically, develop the proof, step by step, calling on this pupil, then that pupil, adding construction lines when necessary at the time they are needed. After the proof has been fully developed, a single pupil may be asked to give it in full; or, the same may be asked the next day, but certainly not at the first reading.

It cannot be too strongly emphasized that real power in geometry comes from practice in mastering original exercises. Hence, for the first ten or twelve weeks the teacher should be satisfied with a very imperfect display of text book knowledge. He should bend his energies toward developing power in mastering originals, even very simple ones; in the end time will be saved, for ability in mastering originals gives a power of comprehension that will make text book theorems easy reading. After a student has discovered himself, so to speak, the most exacting recitation methods may be employed with perfect safety.

Since I have emphasized the importance of exercises, perhaps in order not to be misunderstood, I should say something as a counter-balance. It is a curious fact that in Legendre's geometry, a little less than a hundred years ago, there was not a single exercise; and as late as 1877 Chauvenet's geometry had the following paragraph in the preface: "As the student can make no solid acquisition in geometry without frequent practice in the application of the principles he has acquired, a copious collection of exercises is given in the appendix," etc. Note the word *appendix!* In other words, the only exercises found in the book were at the end. A little later, as you know, the exercises appeared at the end of the various books, or divisions of the texts; and now they are scattered throughout the text. It appears therefore, that exercises came into geometry by the back door and have only recently reached the front door. Will they be content with a supplementary role, or will they, like true revolutionaries, attempt to usurp the whole stage? I hardly think there is danger of the latter contingency, though I do think there is danger of overloading a text with twaddle, so that marily with the rates of such changes and with the determination the fundamentals will not appear in proper relief. It seems to me important that fundamentals and exercises should be shown in their true relations, and that the latter should always be employed to clarify and to aid in gaining a clear understanding of the former. Exercises should, therefore, be significant, they should have a point, and not result in mere busy work. Thus, for example, the insertion of page after page of preliminary work concerning the computation of complimentary and supplementary angles, and of purely mechanical constructions, involving patterns of various designs, etc., is largely a waste of time. The so-called geometers or "rope stretchers of Egypt" constructed right angles, performed computations and probably drew designs for at least twenty centuries without making any appreciable advance in the science of geometry, and it isn't probable that our pupils will profit in any greater degree in two or three weeks' experience with the same kind of material.